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LETTER TO THE EDITOR

2^n-dimensional representations of braid group and link polynomials

Zhong-Qi Ma† and Bao-Heng Zhao‡

† Institute of High Energy Physics, Chinese Academy of Sciences, PO Box 918, Beijing, People's Republic of China

‡ Graduate School, Chinese Academy of Sciences, PO Box 3908, Beijing, People's Republic of China

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Abstract. It is shown that in addition to the Jones polynomial there is another link polynomial obtained from the 2^n-dimensional representation of the braid group which is related to the eight-vertex model. The degeneracy of the Jones polynomial, which Birman pointed out, can be removed partly by the new link polynomial.

Important developments on knot and link theories have recently been made. From the similarity between the Yang–Baxter relation for a solvable model of the statistical mechanics on a two-dimensional lattice (Baxter 1982) and the multiplication rules for the braid group elements, Akutsu and Wadati (1987a, b) found some new representations of the braid group B_n, and then obtained new link polynomials which are more powerful than the well known Alexander polynomial (Alexander 1928) and Jones polynomial (Jones 1985).

In the braid group B_n the generators b_j, $j = 1, 2, \ldots, (n-1)$ satisfy the following multiplication rules:

$$\begin{aligned} &b_j^{-1} b_j = b_j b_j^{-1} = E \\ &b_i b_j = b_j b_i \qquad \text{when } |i-j| \geqslant 2 \\ &b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1}. \end{aligned} \tag{1}$$

Let $D(B_n)$ be a representation of B_n, i.e.

$$b_j \to g_j = D(b_j, n) \tag{2}$$

where g_j satisfies multiplication rules similar to (1).

Tying the top end of each string to the opposite bottom end of a braid (A, n) where A is an element of a braid group B_n, one obtains a closed braid which is a link. We use $L(A, n)$ to denote the closed braid obtained from the braid (A, n). Equivalent links are defined to be links which are topologically equal to each other. Due to the Alexander theorem (Alexander 1923), any oriented link can be represented by a closed braid $L(A, n)$. There are two types of Markov moves (Markov 1935) which relate two braids to the equivalent links:

$$\begin{aligned} &L(AB, n) = L(BA, n) \\ &L(Ab_n^{\pm 1}, n+1) = L(A, n) \end{aligned} \tag{3}$$

where A and B belong to the braid group B_n, and b_n denotes the last generator of the braid group B_{n+1}, $b_n \notin B_n$. An untangled loop (knot) in a link can be continuously deformed to a point. However, according to the usual definition, the link containing an untangled loop is not equal to that which does not contain it, namely

$$L(A, n+1) \neq L(A, n) \qquad A \in B_n \subset B_{n+1}. \tag{4}$$

From a representation of the braid group, a link polynomial might be constructed. All the topologically equivalent links correspond to the same link polynomial. However, some inequivalent links may correspond to the same polynomial. This many-to-one correspondence between links and polynomials is called 'degeneracy'.

If a representation $D(B_n)$ of the braid group B_n is obtained

$$g_j = D(b_j, n) = \mathbb{1}^{(1)} \times \ldots \times \mathbb{1}^{(i-1)} \times \sigma \times \mathbb{1}^{(i+2)} \times \ldots \times \mathbb{1}^{(n)} \tag{5}$$

where $\mathbb{1}^{(k)}$ is an $N \times N$ unit matrix and σ is an $N^2 \times N^2$ matrix satisfying

$$(\sigma \times \mathbb{1})(\mathbb{1} \times \sigma)(\sigma \times \mathbb{1}) = (\mathbb{1} \times \sigma)(\sigma \times \mathbb{1})(\mathbb{1} \times \sigma) \qquad \det \sigma \neq 0 \tag{6}$$

and there exists an $N \times N$ matrix h satisfying

$$\begin{aligned} &[(h \times h), \sigma] = 0 \\ &\sum_q [(\mathbb{1} \times h)\sigma]_{pq,p'q} = \delta_{pp'}\tau \\ &\sum_q [(\mathbb{1} \times h)\sigma^{-1}]_{pq,p'q} = \delta_{pp'}\bar{\tau} \end{aligned} \tag{7}$$

where τ and $\bar{\tau}$ are independent of p, then a link polynomial can be constructed (Akutsu and Wadati 1987b)

$$\begin{aligned} &\alpha(A, n) = (\tau\bar{\tau})^{-(n-1)/2}(\bar{\tau}/\tau)^{e(A)/2} \operatorname{Tr}[HD(A, n)] \\ &H = h \times h \times \ldots \times h. \end{aligned} \tag{8}$$

The link polynomial $\alpha(A, n)$ satisfies the following condition arising from (3):

$$\begin{aligned} &\alpha(AB, n) = \alpha(BA, n) \qquad \alpha(Ab_n^{\pm 1}, n+1) = \alpha(A, n) \\ &A \in B_n \qquad Ab_n^{\pm 1} \in B_{n+1}. \end{aligned} \tag{9}$$

From the exact solutions of N-state vertex models (Sogo *et al* 1983), which are the generalisation of the well known six-vertex model with the Boltzmann weights satisfying $S_{mj}^{lk} = 0$ if $j+k \neq l+m$, Akutsu and Wadati (1987b) obtained some link polynomials with $N = 2, 3, 4$ where

$$\sigma_{jk,lm} = \lim_{u \to \infty} \exp[u(k+m-j-l)/2] S_{mj}^{lk}(u)/S_{ss}^{ss}(u) \tag{10}$$

$$\sigma_{jk,lm} = 0 \qquad \text{if } j+k \neq l+m. \tag{11}$$

The link polynomial with $N = 2$ obtained by Akutsu and Wadati is just the Jones polynomial, which has the degeneracy pointed out by Birman (1985), but the link polynomial with $N = 3$ does not have the Birman degeneracy (Akutsu *et al* 1987).

A more general solvable model is the eight-vertex model, the Boltzmann weights of which satisfy the condition $S_{mj}^{lk} = 0$ if $j+k \neq (l+m) \bmod 2$. In seeking a new representation of the braid group from the eight-vertex model, Akutsu and Wadati met the difficulty of obtaining a definite limit for the Boltzmann weights whilst keeping $S_{-\frac{1}{2}\frac{1}{2}}^{-\frac{1}{2}\frac{1}{2}} \neq 0$.

As a matter of fact, by directly solving (6) and (7) under the condition

$$\sigma_{jk,lm}=0 \qquad \text{if } j+k\neq(l+m) \bmod N \tag{12}$$

the new representation of the braid group and the corresponding link polynomial may be obtained.

For definiteness, in this letter we consider the case $N=2$. Under condition (12), equations (6) and (7) have many solutions, one of which is

$$\sigma=\begin{pmatrix} 0 & 0 & 0 & \sqrt{y} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{y} & 0 & 0 & 0 \end{pmatrix} \qquad h=\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

The other solutions do not give new interesting results. From the solution (13) we obtain a new link polynomial $\alpha(A, n)$

$$\alpha(A, n)=\tfrac{1}{2}\operatorname{Tr} D(A, n) \tag{14}$$

where the representation $D(A, n)$ is generated by g_j in the form (5). Obviously, the link polynomial $\alpha(A, n)$ satisfies condition (9). The Alexander-Conway relation for the polynomial in (14) is

$$\alpha(Ab_j^lB, n)=\alpha(Ab_j^{l-1}B, n)+y\alpha(Ab_j^{l-2}B, n)-y\alpha(Ab_j^{l-3}B, n) \tag{15}$$

where l is an integer, and A, B and b_j belong to B_n.

Birman (1985) has pointed out that the Jones polynomial $\alpha_J(A, n)$ has the following degeneracy:

$$\begin{aligned} &\alpha_J(\Delta^{4m}b_1^{-6m+k}b_2^{-k}, 3)=\alpha_J(b_1^{6m-k}b_2^k, 3) \\ &\Delta=b_1b_2b_1=b_2b_1b_2 \qquad b_1, b_2\in B_3 \end{aligned} \tag{16}$$

for example,

$$\begin{aligned} \alpha_J(\Delta^4b_1^{-12}b_2^6, 3)&=\alpha_J(b_1^{12}b_2^{-6}, 3) \\ &=t^{-3}(1-t+2t^2-3t^3+4t^4-4t^5+6t^6-5t^7+6t^8-6t^9+6t^{10} \\ &\quad -6t^{11}+6t^{12}-5t^{13}+4t^{14}-3t^{15}+2t^{16}-t^{17}+t^{18}). \end{aligned} \tag{17}$$

However, for the link polynomial $\alpha(A, n)$ defined in (14) we have

$$\alpha(\Delta^{4m}b_1^{-6m+k}b_2^{-k}, 3)\neq\alpha(b_1^{6m-k}b_2^k, 3) \tag{18}$$

except for odd k and $k=2m$ or $4m$. For example, the degeneracy for the Jones polynomial in (17) is removed:

$$\begin{aligned} &\alpha(\Delta^4b_1^{-12}b_2^6, 3)=y^{-2}(1+y^2+y^3+y^9) \\ &\alpha(b_1^{12}b_2^{-6}, 3)=y^{-3}(1+y^3+y^6+y^9). \end{aligned} \tag{19}$$

But there are other new degeneracies in $\alpha(A, n)$, for example

$$\alpha(b_1b_2^{2k+1}, 3)=1. \tag{20}$$

If both link polynomials, the Jones polynomial $\alpha_J(A, n)$ and the new polynomial $\alpha(A, n)$, are used to describe a link $L(A, n)$, the Birman degeneracies will be partly removed.

In the following we list some results of the new link polynomials for the closed three-braid $L(A, 3)$:

$$\begin{aligned}
&\alpha(\Delta^{4m} b_1^{2n+1} b_2^{2k+1}, 3) = 1 \\
&\alpha(\Delta^{4m} b_1^{2n+1} b_2^{2k}, 3) = 1 + y^{k+4m} \\
&\alpha(\Delta^{4m} b_1^{2n} b_2^{2k}, 3) = 1 + y^{n+4m} + y^{k+4m} + y^{n+k+4m}
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\alpha(\Delta^{4m} b_1 b_2^{2n+1} b_1 b_2^{2k+1}, 3) = 1 \\
&\alpha(\Delta^{4m} b_1 b_2^{2n+1} b_1 b_2^{2k}, 3) = 1 + y^{k+4m+1} \\
&\alpha(\Delta^{4m} b_1 b_2^{2n} b_1 b_2^{2k}, 3) = 1 + y^{n+4m+1} + y^{k+4m+1} + y^{n+k+4m}
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\alpha(\Delta^{4m} b_1 b_2^{2n} b_1 b_2^{2k} b_1 b_2^{2l}, 3) = 1 + y^{n+k+l+4m} \\
&\alpha(\Delta^{4m} b_1 b_2^{2n} b_1 b_2^{2k} b_1 b_2^{2l+1}, 3) = 1 \\
&\alpha(\Delta^{4m} b_1 b_2^{2n} b_1 b_2^{2k+1} b_1 b_2^{2l+1}, 3) = 1 + y^{k+l+4m+2} \\
&\alpha(\Delta^{4m} b_1 b_2^{2n+1} b_1 b_2^{2k+1} b_1 b_2^{2l+1}, 3) = 1 + y^{n+k+4m+2} + y^{n+l+4m+2} + y^{k+l+4m+2}
\end{aligned} \tag{23}$$

where $m, n, k, l \in \mathbb{Z}$.

Finally, it could be pointed out that the representation (5) of braid group B_n with σ in (13) is equivalent to that with σ_0 satisfying condition (11)

$$\sigma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{y} & 0 \\ 0 & \sqrt{y} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{24}$$

by the similarity transformation X (Akutsu *et al* 1988)

$$\mathrm{X} = \sigma_x \times \mathbb{1} \times \sigma_x \times \mathbb{1} \times \ldots \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{25}$$

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